ON DYNAMICS OF ℓ- VOLTERRA QUADRATIC STOCHASTIC OPERATORS

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Abstract. We introduce a notion of ℓ-Volterra quadratic stochastic operator defined on $(m-1)$-dimensional simplex, where $\ell \in \{0,1,...,m\}$. The ℓ-Volterra operator is a Volterra operator iff $\ell = m$. We study structure of the set of all ℓ-Volterra operators and describe their several fixed and periodic points. For $m = 2$ and 3 we describe behavior of trajectories of $(m-1)$-Volterra operators. The paper also contains many remarks with comparisons of ℓ-Volterra operators and Volterra ones.

Keywords. Quadratic stochastic operator, fixed point, trajectory, Volterra and non-Volterra operators, simplex.

1 Introduction

In biology a quadratic stochastic operator (QSO) has meaning of a population evolution operator (see [15], [16], [17]), which arises as follows. Consider a population consisting of m species. Let $x^0 = (x_1^0, ..., x_m^0)$ be the probability distribution of species in the initial generations, and $P_{ij,k}$ the probability that individuals in the ith and jth species interbreed to produce an individual k. Then the probability distribution $x' = (x_1', ..., x_m')$ (the state) of the species in the first generation can be found by the total probability i.e.

$$x_k' = \sum_{i,j=1}^{m} P_{ij,k} x_i^0 x_j^0, \quad k = 1, ..., m. \tag{1}$$

This means that the association $x^0 \to x'$ defines a map V called the evolution operator. The population evolves by starting from an arbitrary state x^0, then passing to the state $x' = V(x)$ (in the next "generation"), then to the state $x'' = V(V(x))$, and so on. Thus states of the population described by the following dynamical system

$$x^0, \quad x' = V(x), \quad x'' = V^2(x), \quad x''' = V^3(x), ...$$

Note that V (defined by (1)) is a non linear (quadratic) operator, and it is higher dimensional if $m \geq 3$. Higher dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood ([1], [2], [18]).

In this paper we consider a class of nonlinear (quadratic) operators which we call ℓ-Volterra operators and study dynamical systems generated by such operators.

The paper is organized as follows.

In section 2 we give some preliminary definitions. Also we discuss the difference of quadratic operators introduced in this paper from known quadratic operators. In section 3 we describe some invariant (in particular some fixed points) sets for ℓ-Volterra operators. Also we give some family of ℓ-Volterra operators each element of which has cyclic orbits generated by several vertices of the simplex. We also show that the set of all ℓ-Volterra operators is convex, compact and describe its extremal points. Section 4 devoted to 1-Volterra operators and section 5 devoted to 2-Volterra operators defined on a two dimensional simplex. In these sections we describe limit behavior of all trajectories (orbits).

2 Preliminaries

The quadratic stochastic operator (QSO) is a mapping of the simplex.

$$S^{m-1} = \left\{ x = (x_1, ..., x_m) \in \mathbf{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\} \tag{2}$$

into itself , of the form

$$V : x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1, ..., m, \tag{3}$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1, (i, j, k = 1, ..., m), \tag{4}$$

Thus each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbf{P} = (P_{ij,k})_{i,j,k=1}^{n}$ with conditions (4).

Note that each element $x \in S^{m-1}$ is a probability distribution on $E = \{1, ..., m\}$. The population evolves by starting from an arbitrary state (probability distribution on E) $x \in S^{m-1}$ then passing to the state $V(x)$ (in the next "generation"), then to the state $V(V(x)) = V^2(x)$, and so on.

For a given $x^{(0)} \in S^{m-1}$ the trajectory (orbit)

$$\{x^{(n)}\}, \quad n = 0, 1, 2, ... \text{ of } x^{(0)}$$

under the action of QSO (3) is defined by

$$x^{(n+1)} = V(x^{(n)}), \quad \text{where} \quad n = 0, 1, 2, ...$$

One of the main problem in mathematical biology consists in the study of the asymptotical behavior of the trajectories. The difficulty of the problem depends on given matrix **P**. Now we shall briefly describe the history of (particularly) studied QSOs, which allows the reader to easily see the place of the operators introduced in this paper.

The Volterra operators.(see [7], [9],[13]) A Volterra QSO is defined by (3), (4) and the additional assumption

$$P_{ij,k} = 0, \quad \text{if} \quad k \notin \{i, j\}, \ \forall i, j, k \in E. \tag{5}$$

The biological treatment of condition (5) is clear: The offspring repeats the genotype of one of its parents.

In paper [7] the general form of Volterra QSO

$$V : x = (x_1, ..., x_m) \in S^{m-1} \ \rightarrow V(x) = x' = (x'_1, ..., x'_m) \ \in S^{m-1}$$

is given

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right), \tag{6}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \ \text{ for } i \neq k \text{ and } \ a_{ii} = 0, i = 1, ..., m.$$

Moreover

$$a_{ki} = -a_{ik} \ \text{ and } \ |a_{ki}| \leq 1.$$

In [7], [9] the theory of QSO (6) was developed by using theory of the Lyapunov function and tournaments. But non-Volterra QSOs (i.e. which do not satisfy the condition (5)) were not in completely studied. Because there is no any general theory which can be applied for investigation of non-Volterra operators.

To the best of our knowledge, there are few papers devoted to such operators. Now we shall describe non-Volterra operators:

A permutated Volterra operator. Papers [10], [12] are devoted to study of non-Voltera operators which are generated from Volterra operators (6) by a cyclic permutation of coordinates i.e

$$V_\pi : x'_{\pi(i)} = x_i \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right), \quad i = 1, ..., m,$$

where π is a cyclic permutation on the set of indices E.

Kvazi-Volterra QSO. In [4] a class of Kvazi-Volterra operators is introduced. For such operators the condition (5) is not satisfied only for very few values of i, j, k.

Non-Volterra QSO as combination of a Volterra and a non-Volterra operators. In paper [6] it was considered the following family of QSOs $V_\lambda : S^2 \rightarrow S^2$:

$$V_\lambda = (1-\lambda)V_0 + \lambda V_1,\ 0 \le \lambda \le 1,$$

where

$$V_0(x) = \left(x_1^2 + 2x_1x_2,\ x_2^2 + 2x_2x_3,\ x_3^2 + 2x_1x_3\right),$$

is Volterra operator and

$$V_1(x) = \left(x_1^2 + 2x_2x_3,\ x_2^2 + 2x_1x_3,\ x_3^2 + 2x_1x_2\right),$$

is non-Volterra QSO.

Note that behavior of the trajectories of V_0 is very irregular (see [17], [22], [23]).

Non-Volterra QSO generated by a product measure. In [3], [5] a constructive description of the matrix $\mathbf{P}$ is given. This construction depends on a probability measure μ which is given on a fixed graph G and cardinality of a set of cells (configurations).

In [3] it was proven that the QSO constructed by the construction is Volterra iff G is a connected graph.

In [19] using the construction of QSO for the general finite graph and probability measure μ (here μ is product of measures defined on maximal subgraphs of the graph G) a class of non-Volterra QSOs is described.

It was shown that if μ is given by the product of the probability measures then corresponding non-Volterra operators can be studied by N number (where N is the number of maximal connected subgraphs) of Volterra operators defined on the maximal connected subgraphs.

F-QSO. Consider $E_0 = E \bigcup \{0\} = \{0, 1, ..., m\}$. Fix a set $F \subset E$ and call this set the set of "females" and the set $M = E \setminus F$ is called the set of "males". The element 0 will play the role of "empty-body". Coefficients $P_{ij,k}$ of the matrix $\mathbf{P}$ we define as follows

$$P_{ij,k} = \begin{cases} 1, & \text{if } \ k = 0, i, j \in F \cup \{0\} \ \text{ or } \ i, j \in M \cup \{0\}; \\ 0, & \text{if } \ k \neq 0, i, j \in F \cup \{0\} \ \text{ or } \ i, j \in M \cup \{0\}; \\ \ge 0, & \text{if } \ i \in F, j \in M, \forall k. \end{cases} \tag{7}$$

Biological treatment of the coefficients (7) is very clear: a "child" k can be generated if its parents are taken from different classes F and M.

For a given $F \subset E$ a QSO with (3),(4) and (7) is called a F-QSO. Note that F-QSO is non-Volterra for any $F \subset E$.

In [20] the F-QSOs are studied for any $F \subset E$. It was proven that such operators has unique fixed point $(1, 0, ..., 0) \in S^m$ and all trajectories converge to this fixed point faster than any geometric progression.

Strictly non-Volterra QSO. Recently in [21] a new class of non-Volterra operators is introduced. These operators satisfy

$$P_{ij,k} = 0, \quad \text{if} \quad k \in \{i, j\}, \quad \forall i, j, k \in E. \tag{8}$$

Such an operator is called strictly non-Volterra QSO.

For arbitrary strictly non-Volterra QSO defined on S^2 in [21] it was proved that every such an operator has a unique fixed point. Also it was proven that, such operators have a cyclic trajectory. This is quite different behavior from the behavior of Volterra operators, since the Volterra operators have no any cyclic trajectory.

Now we shall give a new class of non-Volterra operators.

ℓ-Volterra QSO. Fix $\ell \in \{1, ..., m\}$ and assume that elements $P_{ij,k}$ of the matrix $\mathbf{P}$ satisfy

$$P_{ij,k} = 0 \ \text{ if } \ k \notin \{i, j\} \quad \text{for any} \quad k \in \{1, ..., \ell\}, \quad i, j \in E; \tag{9}$$

$$P_{ij,k} > 0 \ \text{ for at least one pair } \ (i, j), \quad i \neq k, \quad j \neq k \ \text{ if } \ k \in \{\ell + 1, ..., m\}. \tag{9a}$$

Definition 1. *For any fixed $\ell \in \{1, ..., m\}$, the QSO defined by (3), (4), (9) and (9a) is called ℓ-Volterra QSO.*

Denote by $\mathcal{V}_\ell$ the set of all ℓ-Volterra QSOs.

Remarks. 1. The condition (9a) guarantees that $\mathcal{V}_{\ell_1} \bigcap \mathcal{V}_{\ell_2} = \emptyset$ for any $\ell_1 \neq \ell_2$.

2. Note that ℓ-Volterra QSO is Volterra if and only if $\ell = m$.

3. Kvazi-Volterra operators (introduce above) are particular case of ℓ-Volterra operators.

4. The class of ℓ-Volterra QSO for a given ℓ does not coincide with a class of non-Volterra QSOs mentioned above.

We shall use the following notions.

Definition 2.([1], p. 215)*A fixed point P for $F : \mathbf{R}^m \to \mathbf{R}^m$ is called hyperbolic if the Jacobian matrix $\mathbf{J}$ of the map F at the point P has no eigenvalues on the unit circle.*

There are three types of hyperbolic fixed points :

1. P is an *attracting* fixed point if all of the eigenvalues of $\mathbf{J}(P)$ are less than one in absolute value.

2. P is a *repelling* fixed point if all of the eigenvalues of $\mathbf{J}(P)$ are greater than one in absolute value.

3. P is a *saddle* point otherwise.

The following theorem is also very useful.

Theorem 1. ([1], p.217) *Suppose F has a saddle fixed point P. There exist $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^2$ such that $\gamma(0) = P$; $\gamma'(t) \neq 0$; $\gamma'(0)$ is an unstable eigenvector for $\mathbf{J}(P)$; γ is $F^{-1}-$ invariant; $F^{-n}(\gamma(t)) \to P$ as $n \to \infty$; if $|F^{-n}(q) - P| < \varepsilon$ for all $n \geq 0$ then $q = \gamma(t)$ for some t.*

The curve γ is called the (local) unstable manifold at P. The theorem is true for stable sets as well as with the obvious modification. On the local manifold, all points tend to the fixed point under iteration of F.

3 Canonical form of ℓ-Volterra QSO.

By definition for $k = 1, ..., \ell$ we have

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j = P_{kk,k} x_k^2 + 2 \sum_{\substack{i=1 \\ i \neq k}}^{m} P_{ik,k} x_i x_k =$$

$$x_k \left(P_{kk,k} x_k + 2 \sum_{\substack{i=1 \\ i \neq k}}^{m} P_{ik,k} x_i \right).$$

Using $x_k = 1 - \sum_{\substack{i=1 \\ i \neq k}}^{m} x_i$ we get

$$x'_k = x_k \left(P_{kk,k} + \sum_{\substack{i=1 \\ i \neq k}}^{m} \left(2P_{ik,k} - P_{kk,k} \right) x_i \right), \quad k = 1, ..., \ell.$$

For $k = \ell + 1, ..., m$ we have

$$x'_k = x_k \left(P_{kk,k} + \sum_{\substack{i=1 \\ i \neq k}}^{m} \left(2P_{ik,k} - P_{kk,k} \right) x_i \right) + \sum_{\substack{i,j=1 \\ i \neq k \\ j \neq k}}^{m} P_{ij,k} x_i x_j.$$

Denote $a_{ki} = 2P_{ik,k} - P_{kk,k}$ then

$$\begin{cases} x'_k = x_k \left(a_{kk} + \sum_{\substack{i=1 \\ i \neq k}}^{m} a_{ki} x_i \right), \quad k = 1, ..., \ell \\ x'_k = x_k \left(a_{kk} + \sum_{\substack{i=1 \\ i \neq k}}^{m} a_{ki} x_i \right) + \sum_{\substack{i,j=1 \\ i \neq k \\ j \neq k}}^{m} P_{ij,k} x_i x_j, \quad k = \ell + 1, ..., m. \end{cases} \tag{10}$$

Note that $0 \leq a_{kk} \leq 1$ and $-a_{kk} \leq a_{ki} \leq 2 - a_{kk}$, $i \neq k$, $0 \leq P_{ij,k} \leq 1$.

For any $I \subset E = \{1, 2, ..., m\}$ we define the face of the simplex S^{m-1}:

$$\Gamma_I = \left\{ x \in S^{m-1} : x_i = 0 \ \text{ for any } \ i \in I \right\}.$$

Proposition 1. *Let V be a ℓ-Volterra QSO. Then the following are true*

(i) Any face Γ_I with $I \subseteq \{1, ..., \ell\}$ is invariant set with respect to V.

(ii) Let $A_\ell = \{i \in \{1, ..., \ell\} : a_{ii} > 0\}$. For any $I \subset A_\ell \cup \{\ell + 1, ..., m\}$ the set $T_I = \{x \in S^{m-1} : x_i > 0, \forall i \in I\}$ is invariant with respect to V.

Proof. (i) From (10) it follows that if $x_i = 0$ then $x'_i = 0$ for any $i \in \{1, ..., \ell\}$. Hence $V(\Gamma_I) \subset \Gamma_I$ if $I \subset \{1, ..., \ell\}$.

(ii) Take $I \subset A_\ell \cup \{\ell+1, ..., m\}$. For $k \in I \cap A_\ell$ by (10) and inequality $-a_{kk} \leq a_{kj}, j = 1, ..., m$ we get

$$x'_k = x_k \left(a_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{m} a_{kj} x_j \right) \geq$$

$$x_k \left(a_{kk} - a_{kk} \sum_{\substack{j=1 \\ j\neq k}}^{m} x_j \right) = a_{kk} x_k^2 > 0, \quad \text{since} \quad x_k > 0 \quad \text{for} \quad k \in I \cap A_\ell.$$

For $k \in I \cap \{\ell+1, ..., m\}$ by (10) and condition (9a) we have

$$x'_k = x_k \left(a_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{m} a_{kj} x_j \right) + \sum_{\substack{i,j=1 \\ i\neq k \\ j\neq k}}^{m} P_{ij,k} x_i x_j \geq$$

$$x_k \left(a_{kk} - a_{kk} \sum_{\substack{j=1 \\ j\neq k}}^{m} x_j \right) + \sum_{\substack{i,j\in I \\ i\neq k \\ j\neq k}} P_{ij,k} x_i x_j > a_{kk} x_k^2 \geq 0,$$

here we used $\sum_{\substack{i,j\in I \\ i\neq k \\ j\neq k}} P_{ij,k} x_i x_j > 0$ which follows from condition (9a) and $x_i > 0, x_j > 0, \ \forall i, j \in I$. Thus $V(T_I) \subset T_I$ if $I \subset A_\ell \cup \{\ell+1, ..., m\}$. The proposition is proved.

Denote $e_i = (\delta_{1i}, ..., \delta_{mi}) \in S^{m-1}$, $i = 1, ..., m$ the vertices of the simplex S^{m-1}, where δ_{ij} is the Kronecker's symbol.

Proposition 2. 1) *The vertex e_i is a fixed point for a ℓ-Volterra QSO iff $P_{ii,i} = 1$, $(i = 1, ..., m)$.*

2) *For any collection $I_s = \{e_{i_1}, ..., e_{i_s}\} \subset \{e_{\ell+1}, ..., e_m\}$, $(s \leq m - \ell)$ there exists a family $\mathcal{V}_\ell(I_s) \subset \mathcal{V}_\ell$ such that $\{e_{i_1}, ..., e_{i_s}\}$ is a s-cycle for each $V \in \mathcal{V}_\ell(I_s)$.*

Proof. 1) It is easy to see that if $i \in \{1, ..., \ell\}$ then

$$V(e_i) = (0, ..., 0, P_{ii,i}, 0, ..., 0, P_{ii,\ell+1}, ..., P_{ii,m}) \quad \text{with} \quad P_{ii,i} + \sum_{j=\ell+1}^{m} P_{ii,j} = 1$$

and if $i \in \{\ell+1, ..., m\}$ then

$$V(e_i) = (0, ..., 0, P_{ii,\ell+1}, ..., P_{ii,m}) \quad \text{with} \quad \sum_{j=\ell+1}^{m} P_{ii,j} = 1. \tag{11}$$

Thus $V(e_i) = e_i$ iff $P_{ii,i} = 1$.

2) By (11) we have

$$V(e_{i_j}) = \big(0, ..., 0, P_{i_j i_j,\ell+1}, ..., P_{i_j i_j,m}\big)$$

for any $j = 1, ..., s$. In order to get $V(e_{i_1}) = e_{i_2}$ we assume

$$P_{i_1 i_1,i_2} = 1, \quad P_{i_1 i_1,j} = 0, \quad j \neq i_2. \tag{11_1}$$

Then to get $V(e_{i_2}) = e_{i_3}$ we assume

$$P_{i_2 i_2,i_3} = 1, \quad P_{i_2 i_2,j} = 0, \quad j \neq i_3. \tag{11_2}$$

Similarly to get $V(e_{i_{s-1}}) = e_{i_s}$ we assume

$$P_{i_{s-1} i_{s-1},i_s} = 1, \quad P_{i_{s-1} i_{s-1},j} = 0, \quad j \neq i_s. \tag{11_{s-1}}$$

The last assumption follows from $V(e_{i_s}) = e_{i_1}$ i.e

$$P_{i_s i_s,i_1} = 1, \quad P_{i_s i_s,j} = 0, \quad j \neq i_1. \tag{11_s}$$

Hence $\mathcal{V}_\ell(I_s) = \{V \in \mathcal{V}_\ell : \text{ the coefficients of } V \text{ satisfy } (11_1)-(11_s)\}$. The proposition is proved.

For any set A denote by $|A|$ its cardinality.

The next proposition gives a set of periodic orbits of ℓ-Volterra QSOs.

Proposition 3. *For any $I_1, ..., I_q \subset \{\ell+1, ..., m\}$ such that $I_i \bigcap I_j = \emptyset \ \ (i \neq j, i, j = 1, ..., q)$. There exists a family $\mathcal{V}_\ell(I_1, ..., I_q) \subset \mathcal{V}_\ell$ such that each collection $\{e_i, i \in I_j\}, \ \ j = 1, ..., q$ is a $|I_j|-$ cycle for every $V \in \mathcal{V}_\ell(I_1, ..., I_q)$.*

Proof. Since $I_i \bigcap I_j = \emptyset$, $i \neq j$ the family can be constructed using Proposition 2 i.e. $\mathcal{V}_\ell(I_1, ..., I_q) = \bigcap_{i=1}^q \mathcal{V}_\ell(I_i)$.

Remarks. 1) There is not any ℓ-Volterra operator with a periodic orbit $\{e_{i_1}, ..., e_{i_s}\} \subset \{e_1, ..., e_\ell\}$, $1 < s \leq \ell$.

2) Propositions 2 and 3 show that ℓ-Volterra operators have quite different behavior from the behavior of Volterra operators, since Volterra operators have no cyclic trajectories.

Recall that $\mathcal{V}_\ell$ is the set of all ℓ-Volterra operators defined on S^{m-1}.

Proposition 4. (i) *The set $\mathcal{V}_\ell$ is a convex, compact subset of* $\mathbf{R}^{\frac{m(m-1)(m-\ell+1)}{2}}$.

(ii) *The extremal points of $\mathcal{V}_\ell$ are ℓ-Volterra operators with $P_{ij,k} = 0$ or 1 for any i, j, k i.e.*

$$\mathrm{Extr}(\mathcal{V}_\ell) = \{V \in \mathcal{V}_\ell : \text{the matrix } \ \mathbf{P} \ \text{ of } \ V \ \text{ contains only } \ 0 \ \text{ and } \ 1\}.$$

(iii) *If $\ell = m$ then $|\mathrm{Extr}(\mathcal{V}_\ell)| = 2^{\frac{1}{2}m(m-1)}$; if $\ell \leq m-1$ then*

$$|\mathrm{Extr}(\mathcal{V}_\ell)| = \big(m-\ell\big)^{\frac{1}{2}(m-\ell)(m-\ell+1)}\big(m-\ell+1\big)^{(m-\ell+1)\ell}\big(m-\ell+2\big)^{\frac{1}{2}\ell(\ell-1)}.$$

Proof. (i) Since we have one-to-one correspondence between the set of all QSOs and the set of all cubic matrices $\mathbf{P}$, we can consider a QSO V as a point of $\mathbf{R}^{m(m^2-1)}$. The number $\frac{m(m-1)(m-\ell+1)}{2}$ is the number of independent elements of the matrix $\mathbf{P}$ with the condition (9). Let V_1, V_2 be two ℓ-Volterra QSO i.e $V_1, V_2 \in \mathcal{V}_\ell$. We shall prove that $V = \lambda V_1 + (1-\lambda)V_2 \in \mathcal{V}_\ell$ for any $\lambda \in [0,1]$.

Let $P^{(1)}_{ij,k}$ (resp. $P^{(2)}_{ij,k}$) be coefficients of V_1 (resp. V_2). Then coefficients of V has the form

$$P_{ij,k} = \lambda P^{(1)}_{ij,k} + (1-\lambda)P^{(2)}_{ij,k}. \tag{12}$$

By definition coefficients $P^{(1)}_{ij,k}$ and $P^{(2)}_{ij,k}$ satisfy conditions (9) and (9a). Using (12) it is easy to check that $P_{ij,k}$ also satisfy the condition (9) and (9a).

(ii) Assume $V \in \mathcal{V}_\ell$ with $P_{i_0j_0,k_0} = \alpha \neq 0$ and 1 for some i_0, j_0, k_0. Construct two operators V_q with coefficients $P^{(q)}_{ij,k}$, $q = 1,2$ as following

$$P^{(1)}_{ij,k} = \begin{cases} P_{ij,k} & \text{if } (i,j) \neq (i_0, j_0), \\ 1 & \text{if } (i,j,k) = (i_0, j_0, k_0), \\ 0 & \text{if } (i,j,k) = (i_0, j_0, k), \quad k \neq k_0 \end{cases}$$

$$P^{(2)}_{ij,k} = \begin{cases} P_{ij,k} & \text{if } (i,j) \neq (i_0, j_0), \\ 0 & \text{if } (i,j,k) = (i_0, j_0, k_0), \\ \frac{P_{ij,k}}{1-\alpha} & \text{if } (i,j,k) = (i_0, j_0, k), \quad k \neq k_0. \end{cases}$$

Then

$$\alpha P^{(1)}_{ij,k} + (1-\alpha)P^{(2)}_{ij,k} = \begin{cases} P_{ij,k} & \text{if } (i,j) \neq (i_0, j_0), \\ \alpha = P_{i_0j_0,k_0} & \text{if } (i,j,k) = (i_0, j_0, k_0) \\ P_{ij,k} & \text{if } (i,j,k) = (i_0, j_0, k), \quad k \neq k_0 \end{cases} = P_{ij,k}. \tag{12'}$$

Since $\alpha > 0$, from (12') we get $P_{ij,k} = 0$ if and only if $P^{(1)}_{ij,k} = 0$ and $P^{(2)}_{ij,k} = 0$. This means that V_1 and V_2 are ℓ-Volterra operators. Hence $V = \alpha V_1 + (1-\alpha)V_2$. Thus if $P_{ij,k} \in (0,1)$ for some (i,j,k) then V is not an extremal point. Finally, if $P_{ij,k} = 0$ or 1 for any (i,j,k) then the representation $V = \lambda V_1 + (1-\lambda)V_2$, $0 < \lambda < 1$ is possible only if $V_1 = V_2 = V$.

(iii) In order to compute cardinality of $\mathrm{Extr}(\mathcal{V}_\ell)$ we have to know which elements of the matrix $\mathbf{P}$ can be 1.

Denote $\mathcal{P}_{ij} = (P_{ij,1}, ..., P_{ij,m})^t$ the (i,j)th column of $\mathbf{P}$, where $(i,j) \in \mathcal{K} = \{(i,j) : 1 \leq i \leq j \leq m\}$.

Let $n_0(\mathcal{P}_{ij})$ be the number of elements of $\mathcal{P}_{ij}$ which must be zero by conditions (4), (9), (9a).

Put for $\ell \in \{1, ..., m\}$:

$$\mathcal{A} \equiv \mathcal{A}_{em} = \{(i,j) \in \mathcal{K} : i \leq \ell, j \in \{i\} \cup \{\ell+1, ..., m\}\},$$

$$\mathcal{B} \equiv \mathcal{B}_{em} = \{(i,j) \in \mathcal{K} : i \leq \ell, j \leq \ell, i < j\},$$

$$\mathcal{C} \equiv \mathcal{C}_{em} = \{(i,j) \in \mathcal{K} : \ell < i \leq j\}.$$

Note that $\mathcal{K} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. If $\ell = m$ then $\mathcal{C} = \emptyset$.

It is easy to see that

$$n_0(\mathcal{P}_{ij}) = \begin{cases} \ell - 1 & \text{if} \quad (i,j) \in \mathcal{A} \\ \ell - 2 & \text{if} \quad (i,j) \in \mathcal{B} \\ \ell & \text{if} \quad (i,j) \in \mathcal{C} \end{cases}$$

By condition (4) each column contains unique "1". We have $m - n_0(\mathcal{P}_{ij})$ possibilities to write 1 in the column $(i,j) \in \mathcal{K}$. Thus

$$|\text{Extr}(\mathcal{V}_\ell)| = \prod_{(i,j)\in\mathcal{K}} (m - n_0(\mathcal{P}_{ij})) = (m-\ell+1)^{|\mathcal{A}|}(m-\ell+2)^{|\mathcal{B}|}(m-\ell)^{|\mathcal{C}|}.$$

This with

$$|\mathcal{A}| = (m-\ell+1)\ell, \quad |\mathcal{B}| = \frac{1}{2}(\ell-1)\ell, \quad |\mathcal{C}| = \frac{1}{2}(m-\ell+1)(m-\ell),$$

would yield the formula. The proposition is proved.

For the set $\mathcal{V}$ of all QSOs we have $\mathcal{V} \subset \mathbf{R}^{\frac{m(m^2-1)}{2}}$. Note that $\mathcal{V}$ also is a convex, compact set. Its extremal points also are operators with $P_{ij,k} = 0$ or 1 only. It is easy to see that

$$|\text{Extr}(\mathcal{V}_m)| < |\text{Extr}(\mathcal{V}_{m-1})| < \dots < |\text{Extr}(\mathcal{V}_1)| < |\text{Extr}(\mathcal{V})| = m^{\frac{1}{2}m(m+1)}.$$

For example, if $m = 3$ then

$$|\text{Extr}(\mathcal{V}_3)| = 8, \quad |\text{Extr}(\mathcal{V}_2)| = 48, \quad |\text{Extr}(\mathcal{V}_1)| = 216, \quad |\text{Extr}(\mathcal{V})| = 729.$$

The set $\mathcal{V}$ can be written as $\mathcal{V} = \bigcup_{\ell=0}^{m} \mathcal{V}_\ell$. Here $\mathcal{V}_0$ is the set of " 0-Volterra QSO"s i.e for any $k \in \{1, ..., m\}$ there is at least one pair (i,j) with $i \neq k$ and $j \neq k$ such that $P_{ij,k} > 0$.

As it was mentioned above: $\mathcal{V}_m$ is the set of all Volterra operators and $\mathcal{V}_{\ell_1} \bigcap \mathcal{V}_{\ell_2} = \emptyset$ for any $\ell_1 \neq \ell_2 \in \{0, ..., m\}$.

Thus to study dynamics of QSOs from $\mathcal{V}$ it is enough to study the problem for each $\mathcal{V}_\ell$, $\ell = 0, ..., m$.

In general, the problem of study the behavior of $V \subset \mathcal{V}_\ell$ (for fixed ℓ) is also a difficult problem. So in the next sections we consider the problem for small dimensions (i.e $m = 2, 3$) and $\ell = 1, 2$.

4 Case $m = 2$

In the case $m = 2$ we have only 1-Volterra operator $V : S^1 \to S^1$ such that

$$\begin{cases} x' = ax^2 + 2cxy \\ y' = bx^2 + 2dxy + y^2, \end{cases} \tag{13}$$

where $a, b, c, d \in [0, 1)$ (the case $a = 1$ corresponds to Volterra operator), $a+b = c+d = 1$. Using $x + y = 1$ from (13) we get a dynamical system generated by function $f(x) = (a - 2c)x^2 + 2cx, x \in [0, 1], a \in [0, 1), c \in [0, 1]$. By properties of $f(x)$ one can prove the following

Proposition 5. 1) *If $c \leq \frac{1}{2}$, $\forall a \in [0, 1)$ the operator (13) has unique fixed point $\lambda_0 = (0, 1)$ and for any initial point $\lambda^0 = (x^0, y^0) \in S^1$ the trajectory $\lambda^{(n)}$ goes to λ_0 as $n \to \infty$.*

2) *If $c > \frac{1}{2}$, $\forall a \in [0, 1)$ then (13) has two fixed points $\lambda_0 = (0, 1)$ and $\lambda^* = (\frac{2c-1}{2c-a}, \frac{1-a}{2c-a})$ the point λ_0 is repeller. For any initial point $\lambda^0 \in S^1 \setminus \{\lambda_0\}$ the trajectory $\lambda^{(n)}$ tends to λ^* as $n \to \infty$.*

5 Case $m = 3$

In case $m = 3$ one has two ℓ-Volterra operators (for $\ell = 2$ and 1). Here we shall study the 2-Volterra operators.

Arbitrary 2-Volterra operator (for $m = 3$) has the form :

$$\begin{cases} x' = x(a_1x + 2b_1y + 2c_1z) \\ y' = y(2b_2x + d_1y + 2e_1z) \\ z' = z(2c_2x + 2e_2y + z) + a_2x^2 + 2b_3xy + d_2y_2, \end{cases} \tag{14}$$

where

$$a_1 = P_{11,1}, \quad a_2 = P_{11,3}; \quad b_i = P_{12,i}, i = 1, 2, 3; \quad c_1 = P_{13,1},$$
$$c_2 = P_{13,3}; \quad d_i = P_{22,i}, i = 2, 3; \quad e_i = P_{23,i}, i = 2, 3. \tag{15}$$

To avoid many special cases and complicated formulas we consider the case

$$P_{11,1} = P_{22,2}, \quad P_{13,1} = P_{23,2}, \quad P_{12,1} = P_{12,2}. \tag{16}$$

This corresponds to a symmetric (with respect to permutations of 1 and 2) model.

Using $x + y + z = 1$ and condition (16) the operator (14) can be written as

$$\begin{cases} x' = x(2c + (a - 2c)x + 2(b - c)y) \\ y' = y(2c + 2(b - c)x + (a - 2c)y), \end{cases} \tag{17}$$

where $a = P_{11,1} \in [0, 1)$, $b = P_{12,1} \in [0, \frac{1}{2}]$, $c = P_{13,1} \in [0, 1]$, and $x, y \in [0, 1]$ such that $x + y \leq 1$.

Remark. The case $a = P_{11,1} = P_{22,2} = 1$ corresponds to the Volterra case, so we consider only $a \neq 1$.

Theorem 2. (i) *For $c \leq \frac{1}{2}$ the operator (17) has unique fixed point $\lambda_0 = (0, 0)$ which is global attractive point.*

(ii) *Sets* $M_0 = \{\lambda = (x, y) : x = 0\}$, $M_1 = \{\lambda = (x, y) : y = 0\}$, $M_= = \{\lambda = (x, y) : x = y\}$, $M_> = \{\lambda = (x, y) : x > y\}$, $M_< = \{\lambda = (x, y) : x < y\}$ *are invariant with respect to the operator* (17).

(iii) *For* $c > \frac{1}{2}$, $a \neq 2b$ *the operator (17) has four fixed points* $\lambda_0 = (0, 0)$, $\lambda_1 = \left(0, \frac{2c-1}{2c-a}\right)$, $\lambda_2 = \left(\frac{2c-1}{2c-a}, 0\right)$, $\lambda_3 = \left(\frac{1-2c}{a+b-4c}, \frac{1-2c}{a+2b-4c}\right)$. *Moreover* λ_0 *is repeller and*

$$\lambda_1 \ \ \textit{and} \ \ \lambda_2 \ \ \textit{are} \ \ \begin{cases} \textit{attractive, if} \ \ a > 2b \\ \textit{saddle, if} \ \ a < 2b \end{cases}$$

$$\lambda_3 \ \ \textit{is} \ \ \begin{cases} \textit{attractive, if} \ \ a < 2b \\ \textit{saddle, if} \ \ a > 2b. \end{cases}$$

(iv) *For* $c > \frac{1}{2}$, $a = 2b$ *the operator (17) has a repeller fixed point* $\lambda_0 = (0, 0)$ *and continuum set of fixed points* $F = \{\lambda = (x, y) : x + y = \frac{2c-1}{2(c-b)}\}$. *The following line*

$$I_\nu = \{\lambda = (x, y) : y = \nu x, x \in [0, 1]\}$$

is an invariant set for any $\nu \in [0, \infty)$. *If* $\lambda^0 = (x^0, y^0)$ *is an initial point with* $\frac{y^0}{x^0} = \nu$, $(x^0 \neq 0)$ *then its trajectory* $\lambda^{(n)}$ *goes to* $\overline{\lambda}_\nu = \left(\frac{2c-1}{2(c-b)(1+\nu)}, \frac{(2c-1)\nu}{2(c-b)(1+\nu)}\right) \in I_\nu \bigcap F$ *as* $n \to \infty$, $\nu \in [0, \infty)$, *(if* $x^0 = 0$ *then on invariant set* M_0 *we have* $\lambda^{(n)} \to \lambda_1$*).*

(v) *If* $a < 2b$ *then* M_0 *(resp.* M_1*) is the stable manifold of the saddle point* λ_1 *(resp.* λ_2*). If* $a > 2b$ *then* $M_=$ *is the stable manifold of saddle point* λ_3*. There is an invariant curve* γ *passing through* $\lambda_1, \lambda_2, \lambda_3$ *which is unstable manifold for the saddle points.*

Proof. (i) Clearly $\lambda_0 = (0, 0)$ is a fixed point for (17). Note that the Jacobian of (17) at (0,0) has the form

$$\mathbf{J} = \begin{pmatrix} 2c & 0 \\ 0 & 2c \end{pmatrix},$$

so λ_0 is an attractive if $c < \frac{1}{2}$ and non-hyperbolic if $c = \frac{1}{2}$.

Now we shall prove (for $c \leq \frac{1}{2}$) its global attractiveness. From the first equation of (17) we have

$$x' = x(ax + 2by + 2cz) \leq qx, \tag{18}$$

where $q = \max\{a, 2b, 2c\}$. By definition of the operator (17) and condition $c \leq \frac{1}{2}$ we have $q \leq 1$. Consider two cases:

Case $q < 1$**.** In this case from (18) we get $x_{n+1} \leq qx_n \leq q^n x^0$, where x_n is the first coordinate of the trajectory $\lambda^{(n)} = V^n(l^0) = (x_n, y_n)$ with initial point $\lambda^0 = (x^0, y^0)$. Thus $x_n \to 0$ as $n \to \infty$. By symmetry of x and y we get $y_n \to 0$ as $n \to \infty$.

Case $q = 1$**.** In this case we get $x_{n+1} \leq x_n$, hence

$$\lim_{n\to\infty} x_n = \alpha \geq 0 \ \text{ exists.}$$

Similarly,

$$\lim_{n\to\infty} y_n = \beta \ \text{ also } \ \text{ exists.}$$

Thus the point (α, β) must be a fixed point for the operator (17). Since $\lambda_0 = (0,0)$ is unique fixed point for $c \le \frac{1}{2}$ (we shall prove uniqueness in section (iii) of this proof), we get $(\alpha,\beta) = (0,0)$.

Remark. The argument used in the case $q = 1$ also works for the case $q < 1$. But in the case $q < 1$ we proved that the rate of convergence to λ_0 is faster than q^n.

(ii) Invariance of $M_0, M_=, M_1$ are straightforward. Invariance of $M_<$, $M_>$ follow from the following equality

$$x' - y' = (x-y)(2cz + a(x+y)), \quad \text{where} \quad z = 1 - x - y \ge 0$$

which can be obtained from (17).

(iii) Clearly $\lambda_0 = (0,0)$ is a fixed point independently on parameters a, b, c. To get other fixed points consider several cases:

Case $x = 0, y \neq 0$**:** From the second equation one gets $y = \frac{2c-1}{2c-a}$ which is between 0 and 1 iff $c > \frac{1}{2}$. Thus $\lambda_1 = \big(0, \frac{2c-1}{2c-a}\big)$ is a fixed point.

Case $x \neq 0, y = 0$ is similar to the previous case and gives $\lambda_2 = \big(\frac{2c-1}{2c-a}, 0\big)$.

Case $x \neq 0, y \neq 0$**:** From (17) one gets a system of linear equations, which has unique solution $\lambda_3 = \big(\frac{2c-1}{4c-a-2b}, \frac{2c-1}{4c-a-2b}\big)$ (for $c > \frac{1}{2}, a \neq 2b$). Note that if $c \le \frac{1}{2}$ then there is only λ_0.

To check the type of fixed points consider Jacobian at $\lambda = (x,y)$

$$\mathbf{J}(\lambda) = \mathbf{J}(x,y) = \begin{pmatrix} 2c + 2(a-2c)x + 2(b-c)y & 2(b-c)x \\ 2(b-c)y & 2c + 2(a-2c)y + 2(b-c)x \end{pmatrix}. \tag{19}$$

It is easy to see that the eigenvalues $\mu_1(\lambda), \mu_2(\lambda)$ of (19) at fixed points are

$$\mu_1(\lambda_2) = \mu_2(\lambda_1) = \mu_2(\lambda_3) = 2(1-c) < 1,$$

$$|\mu_1(\lambda_1)| = |\mu_2(\lambda_2)| = \left|\frac{2c(1-a) + 2b(c-1)}{2c-a}\right| = \begin{cases} < 1 & \text{if } \ a > 2b \\ > 1 & \text{if } \ a < 2b \end{cases}$$

$$|\mu_1(\lambda_3)| = \left|\frac{4c(b-1) + 2a(c-1)}{a + 2b - 4c}\right| = \begin{cases} < 1 & \text{if } \ a < 2b \\ > 1 & \text{if } \ a > 2b \end{cases}$$

This completes the proof of (iii).

(iv) For $a = 2b$ the operator (17) has the following form

$$\begin{cases} x' = x(2c + 2(b-c)(x+y)) \\ y' = y(2c + 2(b-c)(x+y)). \end{cases} \tag{20}$$

It is easy to see that $\lambda_0 = (0,0)$ and any point of $F = \Big\{\lambda = (x,y) : x+y = \frac{2c-1}{2(b-c)}\Big\}$ is fixed point if $c > \frac{1}{2}$. Invariance of I_ν follows easily from the following relation $\frac{y'}{x'} = \frac{y}{x} = \nu$. To check $\lambda^{(n)} \to \overline{\lambda}_\nu$ for $\lambda^0 \in I_\nu$, consider restriction of operator (20) on I_ν which is $x' = \varphi(x) = x\big(2c + 2(b-c)(1+\nu)x\big)$. The function φ has two fixed points $x = 0$ and $\overline{x} = \frac{1-2c}{2(b-c)(1+\nu)}$. The point $x = 0$ is repeller and $\overline{x}$ is attractive independently on ν since $\varphi'(\overline{x}) = 2(1-c) < 1$ for $c > \frac{1}{2}$. One can see that $x^* \geq \overline{x}$ where x^* is the critical point i.e $\varphi'(x^*) = 0$. The graphical analysis shows that $\overline{x}$ is the global attractive point on I_ν.

(v) The existence of γ follows from Theorem 1. Other statements of (v) are straightforward. The theorem is proved.

Note that 2-Volterra operator corresponding to (17) has the following form

$$\begin{cases} x' = x(ax + 2by + 2cz) \\ y' = y(2bx + ay + 2cz) \\ z' = 1 - 2c(x+y) - (a-2c)(x^2+y^2) - 4(b-c)xy \end{cases}. \tag{21}$$

Using Theorem 2 one can describe the phase portraits of the trajectories of (21).

Remark. One of the main goal by introducing the notion of ℓ-Volterra operators was to give an example of QSO which has more rich dynamics than Volterra QSO. It is well known [7] that for Volterra operators (see (6)) if $a_{ij} \neq 0 \quad (i \neq j)$ then for any non-fixed initial point λ^0 the set $\omega(\lambda^0)$ of all limit points of the trajectory $\{\lambda^{(n)}\}$ is subset of the boundary of simplex. But in our case Theorem 2 shows that the limit set need not to be subset of the boundary of S^2.

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